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An Application of Model Theory to Semimodules

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Abstract

In this note, we prove that the theory T of cancellative semimodules over a semiring R has the amalgamation property. If R is an entire cancellative zerosumfree semiring, then T has no model-companion. In particular, the theory of commutative additively cancellative monoids forms an example of a non-companionable theory.

1 Purity in Semimodule Theory

For the basic concepts of model theory we refer to [3]. Let us recall that if A and B are L-structures, a homomorphism $f : A \to B$ is said to be pure if for any positive primitive (p.p. for short) formula and any tuple $\bar{a}$ from A, the validity of $\phi(f(\bar{a}))$ in B entails that of $\phi(\bar{a})$ in A [5]. An L-structure A is said to be p.p. normal if it satisfies the following condition: For all p.p. formulas $\phi(\overline{x}, \overline{y})$ of L and all tuples $\overline{a}$, $\overline{b}$ in A, if there is $\overline{c}$ in A such that $A \models \phi(\overline{a}, \overline{c}) \wedge \phi(\overline{b}, \overline{c})$, then for all $\overline{d} \in A$, $A \models \phi(\overline{a}, \overline{d}) \leftrightarrow \phi(\overline{b}, \overline{d})$. In this note we consider the one-sorted first order language L_R of (left) semimodules over a fixed arbitrary semiring R. The axioms for semirings are as for rings with identity with the difference of addition yielding only a commutative monoid, not necessarily an abelian group. We assume that 0 annihilates R and $0 \neq 1$. An $R-$semimodule M is said to be cancellative if every element of M is cancellable. If every element of M has an additive inverse, the semimodule M is called an $R-$module. Throughout this paper, semimodule means left semimodule over R. By homomorphism, we mean an R-homomorphism. One easily obtains the following result.

Lemma 1
Let M be a module over the semiring R. Then M is p.p. normal.

Let R be a semiring and K the class of all semimodules over R. If $M \in K$ and if W is the subsemimodule of $M \times M$ defined by $W = \{(m, m) | m \in M\}$ then W induces an R-congruence relation $\equiv_W$ on $M \times M$, called the Bourne relation, defined by setting $(m, n) \equiv_W (m', n')$ if and only if there exist elements w and w' of W such that $(m, n) + w = (m', n') + w'$. If $(m, n) \in M \times M$ then we write $(m, n)/W$ instead of $(m, n)/\equiv_W$. The factor semimodule $M \times M/\equiv_W$ is denoted by $M \times M/W$. Since for all $(m, n) \in M \times M$ we have $(m, n)/W + (n, m)/W = (0, 0)/W$, then $M \times M/W$ is an R-module. This left R-module, denoted by M^{Δ}, is called the R-module of differences of M. The following facts (in the following Lemma) are essential.

Lemma 2

(1) A subsemimodule of a cancellative semimodule is cancellative.
(2) A pure subsemimodule of a module is a module.

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(3) Given a semimodule M, there is a homomorphism ξ_M of M into M^{Δ}, defined by $\xi_M(m) = (m, 0)/W$.
(4) ξ_M is an embedding if and only if M is cancellative [2].
(5) ξ_M is a pure embedding if and only if M is a module over the semiring R. (The "if" part follows from [2, Prop. 14.1] and [4, Lemma 3.1(b)]).

Let R be a semiring. An R-semimodule M is said to be pure-injective if for every pure embedding $f : A \to B$, any homomorphism $g : A \to M$ can be extended to a homomorphism $h : B \to M$; that is $g = hf$.

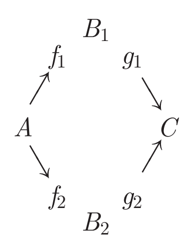
Theorem 3
For any semimodule M over a semiring R there exists a pure-injective R-module N and an R-homomorphism $\alpha_M : M \to N$.

Proof. Let $\xi_M : M \to M^{\Delta}$ be the canonical homomorphism. By Corollary 10.2.2 of [3], the R-module M^{Δ} has an $|L_R|^+$-saturated elementary extension, say, $j_M : M^{\Delta} \to (M^{\Delta})^*$. It follows from Lemma 1 that $(M^{\Delta})^*$ is p.p.normal. Thus, by Theorem 10.7.3 of [3], $(M^{\Delta})^*$ is a pure-injective R-module, and $\alpha_M = j_M \xi_M : M \to (M^{\Delta})^*$ is the desired homomorphism. ■

Corollary 4
Every module over a semiring has a pure-injective elementary extension.

2 Amalgamation property

Let T be a theory (i.e. a deductively closed consistent set of sentences) in a first-order language L. T has the amalgamation property if whenever A, B_1, B_2 are models of T and $f_i : A \to B_i$ are embeddings $(i=1,2)$ then there is a model C of T and embeddings $g_i : B_i \to C$ such that the following diagram is commutative:



A model companion of a theory T is a theory T^* extending T such that T^* is model-consistent with T (i.e. every model of T is embeddable in a model of T^*) and model complete.

For a semiring R, let T be the theory of cancellative R-semimodules.

Theorem 5
T has the amalgamation property.

Proof. Let M, M_1, M_2 be cancellative R-semimodules and $f_i : M \to M_i, (i = 1, 2)$ be embeddings of R-semimodules.

Let $\xi_i = \xi_{M_i} : M_i \to M_i^{\Delta}$, $(i = 1, 2)$, be the canonical homomorphisms. We define a homomorphism $h : M \to M_1^{\Delta} \times M_2^{\Delta}$ by $h(m) = (\xi_1 f_1(m), -\xi_2 f_2(m))$. Let N be the factor semimodule $M_1^{\Delta} \times M_2^{\Delta}/h(M)$. Note that N is cancellative [2].

Consider the homomorphisms

$$\begin{aligned} g_1 : M_1 &\to M_1 \times M_2 \to M_1^{\triangle} \times M_2^{\triangle} \to N \\ x &\to (x, 0) \to (\xi_1(x), 0) \to (\xi_1(x), 0)/h(M), \end{aligned}$$

and

$$\begin{aligned} g_2 : M_2 &\to M_1 \times M_2 \to M_1^{\triangle} \times M_2^{\triangle} \to N \\ y &\to (0, y) \to (0, \xi_2(y)) \to (0, \xi_2(y))/h(M). \end{aligned}$$

We show that $g_i, (i = 1, 2)$, is injective: Suppose, e.g., $g_1(u) = g_1(v)$. Then there are $m_1, m_2 \in M$ such that $(\xi_1(u), 0) + h(m_1) = (\xi_1(v), 0) + h(m_2)$.

Hence $\xi_1(u) + \xi_1 f_1(m_1) = \xi_1(v) + \xi_1 f_1(m_2)$,

$-\xi_2 f_2(m_1) = -\xi_2 f_2(m_2)$.

The injectivity of $\xi_2 f_2$ implies that $m_1 = m_2$. Since M_1 is cancellative, one obtains $u = v$.

Let $m \in M$. One easily checks that

$$(\xi_1 f_1(m), 0) + h(0) = (0, \xi_2 f_2(m)) + h(m).$$

This means that for each $m \in M$,

$$(\xi_1 f_1(m), 0)/h(M) = (0, \xi_2 f_2(m))/h(M)$$

Hence $g_1 f_1(m) = g_2 f_2(m)$, and so $g_1 f_1 = g_2 f_2$. ■

A semiring R is zerosumfree if and only if $r + s = 0$ implies $r = s = 0$, for all $r, s \in R$. A semiring having no nonzero zero divisors is entire. If R is an entire zerosumfree semiring, if M is an R-semimodule and if ∞ is an element not in M then we can define the R-semimodule $M\{\infty\}$ to be the set $M \cup \{\infty\}$ on which the operations of addition and scalar multiplication from M have been extended by setting $x + \infty = \infty + x = \infty$ for all $x \in M\{\infty\}$, $r\infty = \infty$ for all $0 \neq r \in R$ and $0\infty = 0_M$; (cf. [2, Example 13.7).

Lemma 6

If R is an entire zerosumfree semiring then the only pure-injective absolutely pure R-module is $\{0\}$.

Proof. Let M be a pure-injective absolutely pure R-module. Consider the following diagram

$$\begin{array}{ccc} M & \xrightarrow{f} & M\{\infty\} \\ 1_M \downarrow & & \\ M & & \end{array}$$

Since the inclusion mapping f is pure and M is pure-injective, then there exists a homomorphism $g : M\{\infty\} \to M$, such that $g(m) = m$, for all $m \in M$. Let $g(\infty) = y$. For each $m \in M$, we have $m + y = g(m) + g(\infty) = g(m + \infty) = g(\infty) = y$. Hence $m = 0$, and so $M = \{0\}$. ■

Theorem 7

Let R be an entire cancellative zerosumfree semiring and T be the theory of cancellative R-semimodules. Then T has no model-companion.

Proof. We go by contradiction and assume that T has a model-companion T^*. Let E_T be the class of existentially closed models of T. By Proposition 7.10 of [1], E_T is the class of models of T^*. Every object of E_T is, in particular, an absolutely pure R-module. Let $M \in E_T$.

By Corollary 4 , M has a pure-injective elementary extension M^*. Since T is inductive and T^* is a model-companion of T, then by Corollary 7.13 of [1], E_T is an elementary class.

Hence $M^* \in E_T$. It follows immediately from the preceding Lemma, that $M^* = \{0\}$, and so $M = \{0\}$. Since T^* is model-consistent with T, then the only cancellative R-semimodule is $\{0\}$. In particular, ${}_R R = \{0\}$, which is a contradiction. Therefore, T_R has no model-companion. ■

Note that $\mathbb{N}$ is an entire cancellative zerosumfree semiring and the cancellative $\mathbb{N}$-semimodules are precisely the commutative additively cancellative monoids. Thus the following result is then immediate.

Corollary 8
The theory of commutative additively cancellative monoids has no model-companion.

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